AN ENERGY-CONSERVING DIFFERENCE SCHEME FOR THE STORM SURGE EQUATIONS [1]

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ABSTRACT

A system of finite difference equations for storm surge prediction has been constructed, using forward time differences.

The scheme was tested for special simple geometrical configurations, and it was found to be stable without introducing smoothing operators. The variation with time of the total energy was, in each case, the test of stability.

The small-scale oscillation of the energy with time (characteristic of forward difference schemes) was studied in detail. A method of reducing this effect is suggested.

A completely implicit finite difference scheme is discussed from the point of view of stability and convergence. It is shown how the requirement of a convergent iterative process actually introduces a severe restriction on the ratio $\Delta t/\Delta s$, thus canceling the advantages of the otherwise unconditionally stable implicit schemes.

1. INTRODUCTION

A large number of papers dealing with the storm surge problem have already been published. The nonlinearity of the general equations has been eliminated by most research workers, thus facilitating the study of the simplified system, under idealized initial and boundary conditions.

The numerical approach to the remaining linearized equations, however, varies widely. Generally centered differences are used to approximate space and time derivatives. In order to obtain conservation of energy and well behaved fields, various smoothing operators or frictional terms are included in the numerical equations. However, so far, no general criteria have been obtained to the best knowledge of the author.

Such a situation indicates the need of searching for a more suitable finite difference scheme testing its accuracy and efficiency.

In the following paragraphs the computational stability and convergence of a finite difference analog involving the simplest possible mathematical assumptions will be discussed. This will be done by comparing the numerical results to analytical solutions for idealized physical models.

2. THE DIFFERENCE SCHEME

As the principal aim of this paper is the study of the behavior of a numerical solution, and as the proper choice of frictional terms is another problem still unsolved, they will be omitted in the following paragraphs.

The linearized system of equations for the storm surge problem becomes (e.g., Welander [10]):

$$\begin{aligned}\frac{\partial U}{\partial t}&=fV-gh(x,y)\frac{\partial\eta}{\partial x}-\frac{h(x,y)}{\rho}\frac{\partial p^a}{\partial x}\\ \frac{\partial V}{\partial t}&=-fU-gh(x,y)\frac{\partial\eta}{\partial y}-\frac{h(x,y)}{\rho}\frac{\partial p^a}{\partial y}\\ \frac{\partial\eta}{\partial t}&=-\frac{\partial U}{\partial x}-\frac{\partial V}{\partial y}.\end{aligned}\tag{1}$$

Symbols are defined as follows:

- U,V: vertically integrated velocity components in the x and y directions respectively.
- η: elevation of the free surface over the undisturbed level.
- $h(x,y)$: depth of the sea bottom.
- p^a: prescribed atmospheric pressure.
- g: acceleration due to gravity.
- ρ: density of the water.
- f: Coriolis parameter.

System (1) will be expressed in the following difference form:

$$\eta^{n+1}_{J,K}=\eta^n_{J,K}-\frac{\Delta t}{2\Delta s}\left(U^n_{J+1,K}-U^n_{J-1,K}+V^n_{J,K+1}-V^n_{J,K-1}\right)$$

$$U^{n+1}_{J,K}=U^n_{J,K}+f\Delta t V^n_{J,K}-g\frac{\Delta t}{2\Delta s}h_{J,K}$$

$$(\eta^{n+1}_{J+1,K}-\eta^{n+1}_{J-1,K})-\frac{\Delta t}{\rho}h_{J,K}\left(\frac{\partial p^a}{\partial x}\right)^{n+1}_{J,K}$$

[1] This research was performed under U.S. Weather Bureau Contract No. Cwb-11207.

$$V^{n+1}_{J,K}=V^{n}_{J,K}-f\Delta tU^{n+1}_{J,K}-g\frac{\Delta t}{2\Delta s}h_{J,K}(\eta^{n+1}_{J,K+1}-\eta^{n+1}_{J,K-1})-\frac{\Delta t}{\rho}h_{J,K}\left(\frac{\partial p'}{\partial y}\right)^{n+1}_{J,K} \qquad (2)$$

where the indices (J, K, n) denote a point $(x, y, t)=(J\Delta x, K\Delta y, n\Delta t)$ of the discrete grid, and $\Delta x=\Delta y=\Delta s$.

This forward centered scheme mixes explicit and implicit equations, but when evaluated in the order indicated in (2), the system itself becomes explicit and no iterative process is required.

Furthermore, from the computational point of view, economy is achieved by the fact that no storage of old fields is necessary, because after each field has been evaluated its previous values are not used in the remaining equations.

For similar reasons, not all the fields are evaluated at the same points. Transport components are computed at even $(J+K)$ points, and height values at odd $(J+K)$ points, for all n.

By an analysis similar to the one discussed by Platzman [6], the stability condition for system (2) is

$$gh_{max}\frac{\Delta t^2}{\Delta s^2}\leq\frac{4-f^2\Delta t^2}{2-f\Delta t}=2+f\Delta t$$

where h_{max} is the maximum value of the depth.

For all practical purposes this condition reduces to

$$\frac{\Delta t}{\Delta s}\leq\sqrt{\frac{2}{gh_{max}}}. \qquad (3)$$

Due to the fact that forward differences in time are used, no special starting procedures are needed, once the initial state of the fields has been specified.

To simulate a closed region, the transport component normal to the "walls" of the basin has to vanish when working in a rectangular region, e.g.

$$\begin{aligned} U&=0 \quad \text{for} \quad x=0, L_x \\ V&=0 \quad \text{for} \quad y=0, L_y. \end{aligned}$$

These conditions are easily applied to the last two equations of system (2), but some additional procedure must be considered for the evaluation of the centered differences of the transport in the continuity equation.

Welander [10] and Harris and Jelesnianski [2] suggested the addition of auxiliary lines of zero velocity points surrounding the real basin, to be used for the evaluation of η values on the boundaries. However, Harris and Jelesnianski did not actually use this scheme in their calculations, pointing out that the corner points introduced could become source points for small-scale disturbances not germane to the problem.

Their prediction has been verified in this study. It can be seen in figure 1B how the effect of reflection from the boundaries, that occurs earlier (because of the shorter distance) wherever the fictitious "shore points" are not

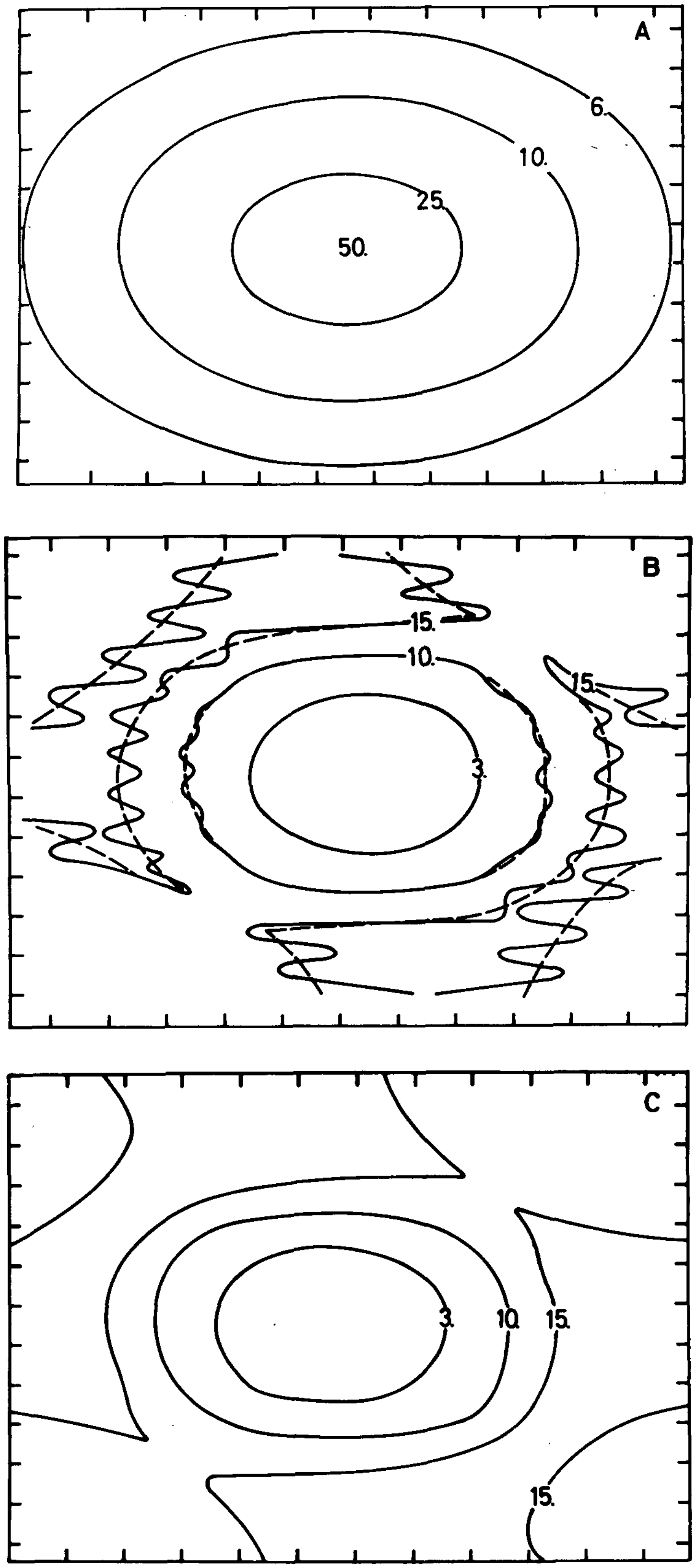

FIGURE 1.—(A) Initial disturbance for a rectangular basin. No forcing functions. The lines are contours of equal height (in cm.). (B) After 12 time steps with auxiliary zero velocity points surrounding the basin next to each η point belonging to the real boundary. (C) After 12 time steps when forward differences were used in the continuity equation for evaluation of η on real boundaries.

utilized, is being propagated into a rectangular basin, with the consequent distortion of the fields. Therefore another method was tried.

Figure 1C shows the free oscillation of the same initial disturbance, after the same number of time steps, where the auxiliary lines were removed and the η values at the corresponding points evaluated by forward differences of transport mesh points lying within the real basin.

Further numerical experiments were carried out in order to study the behavior of this boundary scheme.

In a rectangular region of constant depth, when no forcing functions are acting, and no Coriolis term is considered, an initial sinusoidal disturbance will be propagated through the basin, satisfying the wave equation

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\frac{1}{gh}\frac{\partial^2\eta}{\partial t^2}. \tag{4}$$

The corresponding solution is:

$$\eta(x,\,y,\,t)=\eta_0\cdot\cos\alpha x\cdot\cos\beta y\cdot\cos\sigma t \tag{5}$$

where α and β are

$$\alpha=\frac{m\pi}{L_x},\ \beta=\frac{r\pi}{L_y},\ m,r\ \text{integers},$$

in order to satisfy the boundary conditions

$$\left.\frac{\partial\eta}{\partial x}\right|_{0,L_x}=0,\qquad \left.\frac{\partial\eta}{\partial y}\right|_{0,L_y}=0.$$

The period of oscillation is given by

$$T=\frac{2\pi}{\sigma}=2\bigg/\sqrt{\left(\frac{m^2}{L_x^2}+\frac{r^2}{L_y^2}\right)\cdot gh}. \tag{6}$$

Several cases were solved numerically under the conditions previously specified and for different values of m,r. The initial state was given by (5) with $t=0$.

The resulting fields were behaving according to the analytic solution (5) even after a large number of reflections at the lateral boundaries.

Figure 2 corresponds to mode (1,1). With the time step used in these computations, $T=25.3$ time steps. The numerically obtained values of η, as a function of time at a given point, are plotted. As can be seen the period agrees with the value given by (6).

In order to test the scheme in a case in which one of the boundaries is not a coordinate line, the same problem was solved for a basin in the shape of a right-angle triangle of two equal sides a (fig. 3). In this case an analytic solution exists, to wit:

$$\eta(x,y,t)=\eta_0\cdot\eta_{m,p}(x,y)\cdot\cos\left[\sigma_{m,p}t\right]$$

$$\eta_{m,p}(x,y)=\cos\left[\frac{\pi}{a}\,mx\right]\cdot\cos\left[\frac{\pi}{a}\,(m-p)y\right]$$
$$+(-1)^p\cdot\cos\left[\frac{\pi}{a}\,(m-p)x\right]\cdot\cos\left[\frac{\pi}{a}\,my\right], \tag{7}$$

with

$$\sigma^2_{m,p}=\frac{\pi^2}{a^2}\left[m^2+(m-p)^2\right]\cdot gh. \tag{8}$$

The difficulty that arises in a numerical solution is the application of the condition for zero normal transport

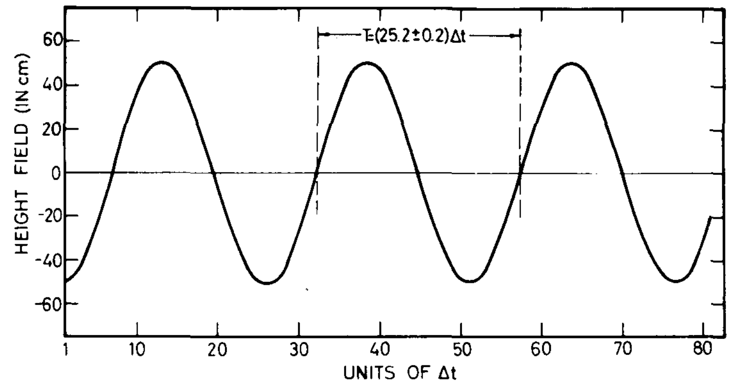

FIGURE 2.—η field at $x=0$, $y=24\ \Delta s$, for a (1,1) mode oscillating freely in a rectangular basin of constant depth. Theoretical period, 25.3 Δt.

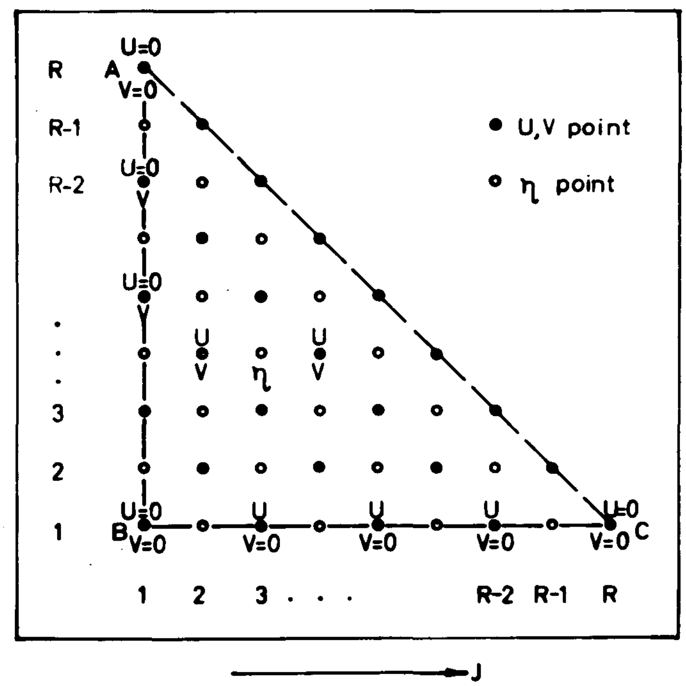

FIGURE 3.—Arrangement of grid points in a triangular basin.

component on the boundary line $\overline{AC}$. This gives a relation between both transport components, which in this case reduces to

$$U+V=0 \tag{9}$$

but not a definite value for either one of them.

In the interior of the basin, and on the lines $\overline{AB}$ and $\overline{BC}$, the usual scheme can be applied. Assuming that all the fields have already been computed at all those points for time $n+1$, the following procedure was used for the boundary line $\overline{AC}$, where the condition (9) is given. The equations for a velocity point of that line were written using forward differences for the space derivatives. Then if $\bar{\eta}$ indicates an η value for the boundary $\overline{AC}$, consisting originally of only velocity points,

$$U^{n+1}_{J,K}=U^n_{J,K}+f\Delta t\,V^n_{J,K}-g\,\frac{\Delta t}{\Delta s}\,h(\bar{\eta}_{J,K}-\eta^{n+1}_{J-1,K}) \tag{10}$$

$$V^{n+1}_{J,K}=V^n_{J,K}-f\Delta t\,U^{n+1}_{J,K}-g\,\frac{\Delta t}{\Delta s}\,h(\eta^{n+1}_{J,K+1}-\bar{\eta}_{J,K})$$

where the indices (J,K) take the values $(J=R+1-K,\,K)$,

R being the number of grid points in the x or y direction. The two equations (10) and the condition (9)

$$U^{n+1}_{J,K}+V^{n+1}_{J,K}=0$$

form a system of three equations and three unknowns. The determinant of the coefficients is different from zero and the system has therefore a nontrivial solution. It provides the velocity values on the boundary $\overline{AC}$, and the auxiliary $\overline{\eta}$ points at time $(n+1)\Delta t$.

This scheme was used in the numerical solution for the case of an initial disturbance

$$\eta(x,y,t=0)=\eta_0\cdot\left[\cos\frac{\pi x}{a}-\cos\frac{\pi y}{a}\right] \tag{11}$$

oscillating freely in a triangular basin of constant depth.

According to (8), the period of this $m=1$, $p=1$ mode corresponds to 42 time steps, with the parameters used in the computation. This was also the value obtained from the numerical computations.

After 126 time steps (3 complete periods) the η field still satisfied the relation (11), with a maximum deviation of 0.2 percent from the initial configuration.

The boundary scheme was therefore considered satisfactory. It is hoped that the method can be generalized for boundaries of irregular shape.

3. THE DIFFERENCE EQUATION FOR THE ENERGY

An accurate solution of the primitive differential equations must conserve energy as long as no external forces are acting on the system. An approximate solution obtained from finite differences should also conserve energy, while nonconservation can indicate an instability in the calculation.

For most centered difference schemes one can define the energy integral in such a way that conservation of energy is automatically fulfilled as long as the calculation is stable.

The main objection (see for example Harris and Jelesnianski [3]) against non-time-centered schemes like (2) is that there is no clear way of defining and calculating the energy balance. This is due to the fact that even though each field can be labelled with the same time step index (as in (2)), they actually represent the situation of the system at slightly different times. Therefore, it was considered desirable to go more deeply into this problem, and to try to suppress or estimate those spurious variations of the total energy, which although inherent in the scheme, do not indicate any instability.

The energy relation corresponding to system (1) is:

$$\frac{\partial}{\partial t}\iint_S\left(\frac{U^2+V^2}{2h(x,y)}+\frac{g}{2}\eta^2\right)dxdy+\frac{1}{\rho}\iint_S p^a\frac{\partial\eta}{\partial t}dxdy$$
$$+\oint_C\left(g\eta+\frac{1}{\rho}p^a\right)\mathbf{M}\cdot\mathbf{n}dl=0,\quad \mathbf{M}=U\mathbf{i}+V\mathbf{j} \tag{12}$$

where S is the surface area of the basin and C its contour. The last term, involving the normal transport components to the boundaries, will cancel in the case of a closed basin.

Since the transport and η fields are evaluated only at discrete points, the energy balance must be solved by numerical quadrature. With the staggered grid used in the computations, it seems natural to replace the integrals by summation over elements of triangular shape covering the whole basin. By such a procedure, at any given time:

$$\frac{2}{(\Delta s^2/2)}\iint_S\frac{U^2}{2h(x,y)}dxdy\simeq\left(\frac{U^2_{1,1}}{h_{1,1}}+\frac{U^2_{1,T}}{h_{1,T}}+\frac{U^2_{R,1}}{h_{T,1}}+\frac{U^2_{R,T}}{h_{R,T}}\right)$$
$$+2\sum_{K=1}^{(T-3)/2}\left(\frac{U^2_{1,2K+1}}{h_{1,2K+1}}+\frac{U^2_{R,2K+1}}{h_{R,2K+1}}\right)$$
$$+2\sum_{J=1}^{(R-3)/2}\left(\frac{U^2_{2J+1,1}}{h_{2J+1,1}}+\frac{U^2_{2J+1,T}}{h_{2J+1,T}}\right)$$
$$+4\sum_{J=1}^{(R-3)/2}\sum_{K=1}^{(T-3)/2}\left(\frac{U^2_{2J+1,2K+1}}{h_{2J+1,2K+1}}\right)$$
$$+4\sum_{J=1}^{(R-1)/2}\sum_{K=1}^{(T-1)/2}\left(\frac{U^2_{2J,2K}}{h_{2J,2K}}\right), \tag{13}$$

and similarly for the other terms. The length of the basin is given by $(R-1)\Delta s$ and $(T-1)\Delta s$ in the x and y directions respectively.

The computational difficulty arises in the evaluation of the time derivative. In the finite difference analog (2) forward differences were used. On the right hand side of these equations there appear terms evaluated at time n and $n+1$, and the arrangement differs from equation to equation. However, the contribution of each field should be consistently centered in time so as to obtain a value for the energy that will represent the shape of the system at a unique and well defined time. By comparison of (12) with the analogous invariant that can be deduced from the difference equations (2), the following scheme was adopted:

$$(\text{EN})^n=\frac{3\,(\text{EKIN})^{n-1}+(\text{EKIN})^n}{4}+\frac{(\text{EPOT})^{n-1}+3\,(\text{EPOT})^n}{4} \tag{14}$$

where $(\text{EN})^n$ represents the sum of potential plus kinetic energy as assigned to time step n, and

$$\text{EKIN}=\frac{1}{2}\sum_J\sum_K\left(\frac{U^2_{J,K}+V^2_{J,K}}{h_{J,K}}\right)\Delta s^2$$

$$\text{EPOT}=\frac{g}{2}\sum_J\sum_K(\eta^2_{J,K})\Delta s^2.$$

The expression $\sum_J\sum_K$ denotes the summation procedure indicated by (13). The upper index in expression (14) indicates the time step. In addition

$$\text{WORK}=\frac{1}{\rho}\iint p^a\frac{\partial\eta}{\partial t}dxdy$$
$$\simeq\frac{1}{\rho}\sum_J\sum_K(p^a)^{n+1/2}(\eta^{n+1}_{J,K}-\eta^n_{J,K})\frac{\Delta s^2}{\Delta t}.$$

(where $(p^a)^{n+1/2}$ is interpolated from the corresponding values at n and $n+1$). It must be stressed that procedure (14) does not represent any form of smoothing. It is purely an attempt to combine the transport and η fields in such a way as to eliminate or decrease the difference of phase inherent to formulation (2).

The effect of the adoption of scheme (14) on the behavior of the energy can be studied by applying it to a very simplified model. A one dimensional channel of constant depth is considered, in which the fluid is oscillating freely.

The differential system of equations reduces to:

$$\frac{\partial \eta}{\partial t}=-\frac{\partial U}{\partial x}$$

$$\frac{\partial U}{\partial t}=-gh\frac{\partial \eta}{\partial x}.$$

The finite difference analog similar to (2) applied to this system gives:

$$\eta_J^{n+1}=\eta_J^n-\frac{\Delta t}{2\Delta s}\left(U_{J+1}^n-U_{J-1}^n\right)$$

$$U_J^{n+1}=U_J^n-gh\frac{\Delta t}{2\Delta s}\left(\eta_{J+1}^{n+1}-\eta_{J-1}^{n+1}\right). \quad (15)$$

The solution of which can be written in the form:

$$\eta_J^n=N_0\left(\cos n\phi+\frac{\mu}{\sqrt{4-\mu^2}}\sin n\phi\right)\cos\alpha\, J\Delta s$$

$$U_J^n=\frac{2N_0}{\sqrt{4-\mu^2}}\sin n\phi \sin\alpha\, J\Delta s, \quad (16)$$

where

$$\mu=\frac{\Delta t}{\Delta s}\sqrt{gh}\sin\alpha\Delta s;\ \phi=\text{arc cos}\left(1-\tfrac{1}{2}\mu^2\right);$$

$$\alpha=\frac{m\pi}{L}\ (m \text{ an integer}).$$

The initial conditions assumed for this system are:

$$U_J(t=0)=0,$$

$$\eta_J(t=0)=N_0\cos\alpha\, J\Delta s.$$

Equations (16) can be used for the evaluation of the total energy, according with its definition (14). After some algebraic manipulation the expression obtained is:

$$(\text{EN})^n=\frac{\Delta s^2}{2}gC\frac{N_0^2}{4-\mu^2}\left\{4+\frac{\mu^3}{4}\sqrt{4-\mu^2}(\mu^2-1)\sin 2n\phi+\frac{\mu^4}{4}(3-\mu^2)\cos 2n\phi\right\}. \quad (17)$$

C is a constant resulting from the summation implied in (14) and that depends on the total number of grid points in the grid.

It can be seen from expression (17) that oscillations of period $T=\pi\Delta t/\phi$ appear superimposed over a constant value for the energy. If (14) is an accurate representation of the energy, the amplitude of these oscillations should be, of course, very small or zero. Figure 4 describes the behavior of these spurious oscillations. Their amplitude and period depend on μ, which itself is determined by the characteristic wavelength of the imposed motion and the dimensions of the channel. In terms of the numerical scheme, this means that μ is fixed by the number of grid points per wave as used in the computations.

From figure 4 it may be deduced that the amplitude of the "noise" decreases, and its period increases, with the number of grid points per wave, and that its influence is negligible (under 2 percent) when each wave is resolved by at least 8 grid points. The oscillations become important when less than 6 points are assigned to each wave; but this is in any case the critical number of grid points that is necessary to determine a wave [4]. Waves shorter than 5 Δs are not properly resolved by numerical schemes in general.

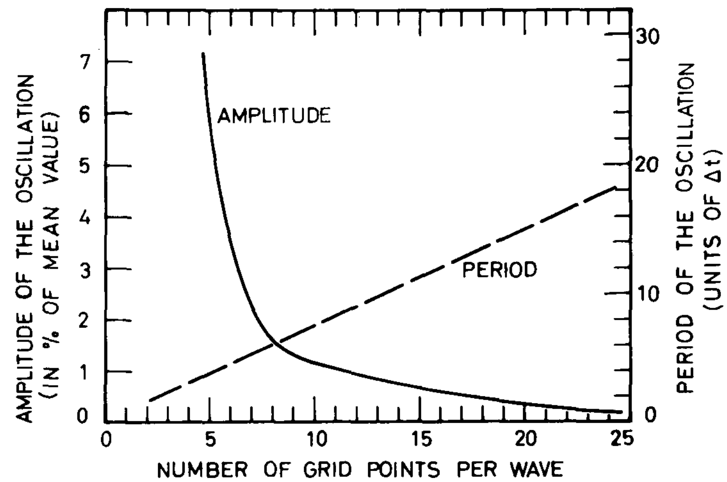

FIGURE 4.—Behavior of the time dependent oscillations superimposed on the energy defined by scheme [14] as a function of the number of grid points per wave in the discrete system.

The accuracy of the value predicted by expression (17) can be checked by treating numerically the same physical system for which this formula was obtained. Figure 5, for example, represents the total energy of such a system, obtained by the numerical solution of equations (15), when the number of grid points per wave was 8. The periodic character of the oscillation of the total energy is evident. Its amplitude is 1.9 percent as compared with the value of 1.6 percent given by (17) (see fig. 4). The period, 6.1 steps, is exactly predicted. A similar good agreement is obtained for other values of the parameters shown in figure 4.

As pointed out by Harris and Jelesnianski [3], the critical test of the stability of computation in the storm surge problem is conservation of energy. The present analysis shows the behavior of a procedure like (14), that involves a correction of the time shift between kinetic and potential energy when forward differences in time are used for the numerical scheme. An easy analysis of the energy balance is therefore possible, and the main objection against the use of forward differences, as in Fischer's [1] scheme, or in the one presented in this paper, is removed.

4. CONVERGENCE

A first check of convergence of the numerical scheme to the solution of the differential equations was already shown in section 2, when the results of free oscillations in a rectangular and triangular basin of constant depth were discussed. There, the model reproduced the analytical solution of the wave equation to a high degree of accuracy.

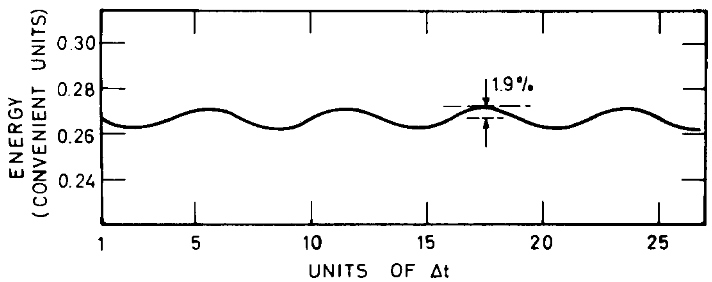

FIGURE 5.—Total energy of a rectangular basin of constant depth oscillating freely with 8 grid points per wave in the x-direction. (All variables independent of y).

In order to test the behavior of the scheme for a system under the action of an external forcing function, a one dimensional solution given by Proudman [7] was chosen.[2]

The solution of the equations of motion in the case of a semi-infinite channel of constant depth is given by

$$\eta(y,\,t)=M\left\{\frac{V_p}{U_c}f\left(t-\frac{y}{U_c}\right)-f\left(t-\frac{y}{V_p}\right)\right\}$$

$$V(y,\,t)=M\left(\frac{V_p}{U_c}\right)\left\{f\left(t-\frac{y}{U_c}\right)-f\left(t-\frac{y}{V_p}\right)\right\} \tag{18}$$

where

$$U_c=\sqrt{gh}, \qquad M=1\bigg/\left\{1-\left(\frac{V_p}{U_c}\right)^2\right\},$$

and $f(t-y/V_p)$, is any function that represents a disturbance of atmospheric pressure advancing at a constant velocity V_p. $f(t-y/U_c)$ is the same function of the argument $(t-y/U_c)$. The coordinate y is taken along the length of the channel. For the numerical experiment the following expression was used

$$f\left(t-\frac{y}{V_p}\right)=\frac{\Delta p}{2}\left\{1-\tanh\left[k\left(\frac{2V_p}{L_p}\left(t-\frac{y+y_0}{V_p}\right)-1\right)\right]\right\}. \tag{19}$$

y_0 is the initial position of the "pressure front" relative to the closed boundary of the channel. k is a parameter that determines the steepness of the hyperbolic tangent curve. A value of this slope which is too high is liable to excite parasitic waves, discrete systems being unable to represent sharp gradients, which involve relatively large amplitudes of short wave components. Several trials lead to $k=1.5$ and a width of the pressure step, $L_p=8\Delta s$.

Both numerical and analytical solutions were obtained for the same values of the parameters. To simulate a semi-infinite channel with the computational scheme, a long and narrow basin was considered, with no variation of variables along the x-direction. The pressure front advanced towards the basin through the closed boundary at $y=0$, with a velocity $V_p=14$ m./sec. and the computations were terminated before reflections from the second boundary (located at infinity in the analytical model) could be propagated into the region of interest.

[2] This same solution was already used by Platzman [5] for preliminary computations related with his work on Lake Michigan.

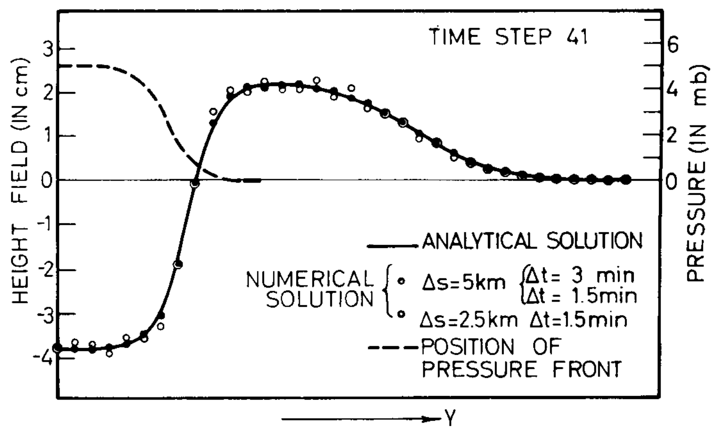

FIGURE 6.—η field after 41 time steps of integration. The dotted line indicates the position of the pressure front over the channel at the time under consideration.

Figure 6 shows the final results. The numerical experiment was carried out for three different values of increments: Δs and the corresponding maximum value of Δt resulting from the stability condition; Δs and $\Delta t/2$, and $\Delta s/2$ and $\Delta t/2$, in order to estimate the dependence of the truncation error on the space and time intervals. The convergence of numerical results to the analytical solution is encouraging.

A variety of other cases, for which no analytical solutions are available, were also studied. Among them: basins of linear, parabolic, or exponential bottom topography. The water was oscillating freely or under the action of wind or pressure, which constituted the external forcing function. In such situations the actual test on the stability of the numerical scheme was the conservation of the energy or its balance by the work exerted on the open system. In this sense, also for these cases scheme (2) proved to be stable.

5. CONCLUSIONS

A careful study of the stability and convergence of the finite difference analog (2) shows that smoothing of the fields is not necessary when such a scheme is used.

The discussion of section 3 indicates a way of evaluating a representative energy of the system when velocities and heights are computed with a lag in time, removing therefore the only serious objection for the use of forward differences in the numerical solution of storm surges problems.

The considerable saving of computing time in the use of the scheme here presented can be an important factor for its adoption in numerical research or in actual forecast.

APPENDIX

AN IMPLICIT DIFFERENCE SCHEME FOR THE PRIMITIVE EQUATIONS OF MOTION

Forward differences in time, and centered differences in space can be applied to system (1) in the following way:

$$U^{n+1}_{J,K}=U^{n}_{J,K}-\sqrt{gh}\,\frac{\Delta t}{2\Delta s}\,\{\theta(\phi^{n+1}_{J+1,K}-\phi^{n+1}_{J-1,K})+(1-\theta)$$
$$\cdot(\phi^{n}_{J+1,K}-\phi^{n}_{J-1,K})\}+f\Delta t\{\theta V^{n+1}_{J,K}+(1-\theta)V^{n}_{J,K}\}$$

$$V^{n+1}_{J,K}=V^n_{J,K}-\sqrt{gh}\,\frac{\Delta t}{2\Delta s}\,\{\theta(\phi^{n+1}_{J,K+1}-\phi^{n+1}_{J,K-1})+(1-\theta)\cdot(\phi^n_{J,K+1}-\phi^n_{J,K-1})\}-f\Delta t\{\theta U^{n+1}_{J,K}+(1-\theta)U^n_{J,K}\} \tag{A1}$$

$$\phi^{n+1}_{J,K}=\phi^n_{J,K}-\sqrt{gh}\,\frac{\Delta t}{2\Delta s}\,\{\theta(U^{n+1}_{J+1,K}-U^{n+1}_{J-1,K})+(1-\theta)(U^n_{J+1,K}-U^n_{J-1,K})+\theta(V^{n+1}_{J,K+1}-V^{n+1}_{J,K-1})+(1-\theta)(V^n_{J,K+1}-V^n_{J,K-1})\}$$

where all the symbols have the same meaning as before, and

$$\phi_{J,K}=\sqrt{gh}\eta_{J,K}.$$

θ is a parameter such that $0\leq\theta\leq 1$.

An analysis of the stability of such a system shows that when $0\leq\theta<\frac{1}{2}$, the numerical scheme is unstable, and if $\frac{1}{2}\leq\theta\leq 1$, the numerical scheme is unconditionally stable.

Therefore, with $\theta=1$ this finite difference analog will have the interesting property that Δt is not limited in size by stability considerations (involving Δs), and only by desired control of truncation error.[3] However, since the system is implicit, some iteration process must be used for the numerical integration of the difference equations.

It is useful to consider the differences between two successive iterations:

$$\begin{aligned} {}^{(i)}\Delta\phi &= {}^{(i)}\phi^{n+1}-{}^{(i-1)}\phi^{n+1}\\ {}^{(i)}\Delta U &= {}^{(i)}U^{n+1}-{}^{(i-1)}U^{n+1}\\ {}^{(i)}\Delta V &= {}^{(i)}V^{n+1}-{}^{(i-1)}V^{n+1}\end{aligned} \tag{A2}$$

with the left upper indices indicating the sequence of iteration steps. Then, if each field is expressed by one of its Fourier components, for every iterative procedure it will be possible to write equation (A2) in the following brief notation:

$$\mathbf{B}\cdot\begin{pmatrix}{}^{(i)}\Delta\tilde{\phi}\\ {}^{(i)}\Delta\tilde{U}\\ {}^{(i)}\Delta\tilde{V}\end{pmatrix}=\mathbf{A}\cdot\begin{pmatrix}{}^{(i-1)}\Delta\tilde{\phi}\\ {}^{(i-1)}\Delta\tilde{U}\\ {}^{(i-1)}\Delta\tilde{V}\end{pmatrix}$$

B and **A** being matrices whose elements depend on the particular iteration technique adopted. The symbol $\sim$ over each field indicates the time dependent amplitude of the corresponding Fourier component. It follows from definition (A2) that if the iteration process is convergent, the differences should monotonically decrease for (i) bigger than a certain value $i=I$; and if Richtmyer's definitions [8] of the bound of a matrix are adopted, the convergence criterion will be

$$\text{Bound } \mathbf{B}^{-1}\mathbf{A}=\underset{|\mathbf{v}|=1}{\text{Max}}\ \mathbf{v}^*(\mathbf{B}^{-1}\mathbf{A})^*(\mathbf{B}^{-1}\mathbf{A})\mathbf{v}<1. \tag{A3}$$

With the aid of this inequality the convergence of some iteration procedures was studied, but for none of them was there found a condition less restrictive than the stability criterion obtained in the case of equation (2).

$$\frac{\Delta t}{\Delta s}\leq\sqrt{\frac{2}{gh}}.$$

These conclusions were verified with a FORTRAN program written to solve equations (A1), with $\theta=1$ and various iterations processes. Iterations imply in any case additional computer time. Therefore implicit systems are not practical, if such a bound must be applied to Δt.

It should be noticed that in the previously mentioned paper by Uusitalo [9] the iteration scheme is not discussed, but the use of the Courant-Friedrichs-Lewy criterion probably avoids the appearance of the convergence problem.

[3] It should be mentioned that the same difference scheme, when $\theta=\frac{1}{2}$, was already used by Uusitalo [9] in his storm surge computations for the North Sea. In this case, the author considered it necessary to include a Laplacian smoothing operator.

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